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COMMENT

Phase diagram of the N-colour Ashkin–Teller model

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Abstract. We show that the two-dimensional N-colour Ashkin-Teller model exhibits an Ising-like continuous transition line in the region where the model has two phase transitions. The result is also valid for the antiferromagnetic ordering. In addition we present results which support the conjecture of Grest and Widom about the location of the disorder point.

The two-dimensional N-colour Ashkin-Teller model has received considerable attention in the past few years. The model Hamiltonian

$$\frac{-\mathcal{H}}{kT}=\sum_{r}\Bigg(\sum_{\alpha=1}^{N}[K^{x}\sigma_{\alpha}(\boldsymbol{r})\sigma_{\alpha}(\boldsymbol{r}+\hat{x})+K^{\tau}\sigma_{\alpha}(\boldsymbol{r})\sigma_{\alpha}(\boldsymbol{r}+\hat{\tau})]$$
$$+\tfrac{1}{2}\sum_{\alpha\neq\beta}[K_{4}^{x}\sigma_{\alpha}(\boldsymbol{r})\sigma_{\alpha}(\boldsymbol{r}+\hat{x})\sigma_{\beta}(\boldsymbol{r})\sigma_{\beta}(\boldsymbol{r}+\hat{x})$$
$$+K_{4}^{\tau}\sigma_{\alpha}(\boldsymbol{r})\sigma_{\alpha}(\boldsymbol{r}+\hat{\tau})\sigma_{\beta}(\boldsymbol{r})\sigma_{\beta}(\boldsymbol{r}+\hat{\tau})]\Bigg) \qquad (1)$$

consists of N Ising models coupled pairwise by a four-spin interaction (K_4), which is able to change drastically the critical behaviour of the system. Because of its relationship to an exactly solved vertex model (Baxter 1971, Wegner 1972) the case $N=2$ is nowadays well known and it has been used as a guide to study higher values of N. In this sense the mapping of the two-colour model onto a Thirring Hamiltonian (Drugowich de Felício and Köberle 1982) has been extended by Shankar (1985), who found in the Gross-Neveu model the field theoretical counterpart of the N-colour Ashkin-Teller one. This connection between those two models led to the proof of the existence of a first-order transition, for $K_4>0$, for all $N\geqslant 3$, a result first quoted by Grest and Widom (1981) after numerical simulations and renormalisation group analyses. The sketch of the phase diagram for a general $N(\geqslant 3)$ is shown in figure 1 where the remarkable features which we are interested in are shown, namely:

(i) a continuous transition line for $\Delta>\Delta_B$ where B is the bifurcation point, and

(ii) a disorder point located at $\Delta=-1/(N-1)$ according to the numerical simulations of Grest and Widom.

The aim of this comment is to prove that, independently of N, the model exhibits Ising-like critical behaviour along the BC lines. The fundamental assumption we make is that the continuous time Hamiltonian, introduced by Fradkin and Susskind (1978), lies in the same universality class, i.e. has the same long-distance behaviour in the

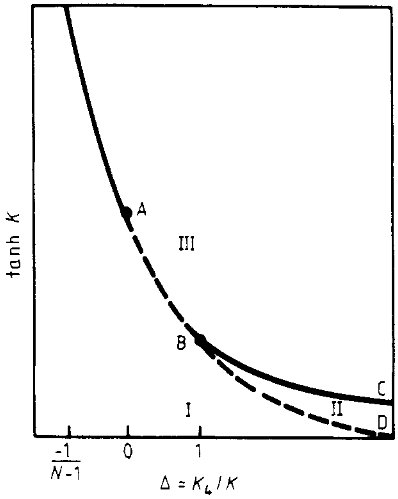

Figure 1. Phase diagram of the isotropic N-colour Ashkin-Teller model. A is the decoupling point which separates first-order (broken curve) and second-order (full curve) phase transitions. In phase I, $\langle\sigma_\alpha\rangle = 0$; in phase III, $\langle\sigma_\alpha\rangle \neq 0$; in phase II, $\langle\sigma_\alpha\rangle = 0$ but $\langle\sigma_1\sigma_\alpha\rangle \neq 0$. At $\Delta = -1/(N-1)$ the model is paramagnetic for any value of K. The critical line BC, separating phases II and III, belongs to the Ising universality class.

critical region, as the original lattice model. In addition, we give arguments which support the Grest-Widom conjecture about the location of the disorder point. The procedure is very simple and it is based on a transfer-matrix analysis. We begin by introducing new variables S_α, given by

$$S_\alpha(\boldsymbol{r}) = \sigma_1(\boldsymbol{r})\sigma_\alpha(\boldsymbol{r}) \qquad \alpha = 2, 3, \ldots, N \tag{2}$$

in terms of which the original action becomes

$$\begin{aligned}\frac{-\mathcal{H}}{kT} = \sum_{\boldsymbol{r}}\Bigg(\sum_{\alpha=2}^{N} & [K^x\sigma_1(\boldsymbol{r})\sigma_1(\boldsymbol{r}+\hat{x})S_\alpha(\boldsymbol{r})S_\alpha(\boldsymbol{r}+\hat{x}) + K^\tau\sigma_1(\boldsymbol{r})\sigma_1(\boldsymbol{r}+\hat{\tau})S_\alpha(\boldsymbol{r})S_\alpha(\boldsymbol{r}+\hat{\tau})] \\ & + \tfrac{1}{2}\sum_{(\alpha\neq\beta)\geqslant 2}[K_4^x S_\alpha(\boldsymbol{r})S_\alpha(\boldsymbol{r}+\hat{x})S_\beta(\boldsymbol{r})S_\beta(\boldsymbol{r}+\hat{x}) \\ & + K_4^\tau S_\alpha(\boldsymbol{r})S_\alpha(\boldsymbol{r}+\hat{\tau})S_\beta(\boldsymbol{r})S_\beta(\boldsymbol{r}+\hat{\tau})] \\ & + \sum_{\alpha=2}^{N}[K_4^x S_\alpha(\boldsymbol{r})S_\alpha(\boldsymbol{r}+\hat{x}) + K_4^\tau S_\alpha(\boldsymbol{r})S_\alpha(\boldsymbol{r}+\hat{\tau})] \\ & + K^x\sigma_1(\boldsymbol{r})\sigma_1(\boldsymbol{r}+\hat{x}) + K^\tau\sigma_1(\boldsymbol{r})\sigma_1(\boldsymbol{r}+\hat{\tau})\Bigg).\end{aligned} \tag{3}$$

Next we perform a highly anisotropic limit by taking

$$K^x, K_4^x \to 0 \tag{4a}$$

and

$$K^\tau, K_4^\tau \to \infty \tag{4b}$$

subject to the condition

$$K_4^x/K^x = K_4^\tau/K^\tau = \Delta. \tag{5}$$

Following the above prescription, which has proved to be effective in describing the phase transitions of the $N=2$ case (Drugowich de Felício *et al* 1982), the transfer matrix can be written as

$$\hat{T} = 1 - \tau \hat{H} \tag{6}$$

with

$$\tau = \exp(-2NK^{\tau}) \qquad K^{x} = \lambda\tau \tag{7}$$

and

$$-\hat{H} = \sum_{j} \bigg(\lambda\sigma_1^z(j)\sigma_1^z(j+1) + \sigma_1^x(j) + \sum_{\alpha=2}^{N} \lambda S_\alpha^z(j) S_\alpha^z(j+1)[\Delta + \sigma_1^z(j)\sigma_1^z(j+1)] + \tfrac{1}{2}\lambda\Delta \sum_{\alpha\neq\beta>2} S_\alpha^z(j)S_\alpha^z(j+1)S_\beta^z(j)S_\beta^z(j+1)\bigg). \tag{8}$$

So, in the critical region, the thermodynamics of the N-colour AT model is governed by the ground-state properties of the Hamiltonian (8), whose minimum is attained, if $\Delta > 1$, for a ferromagnetic ordering of the classical variables S, i.e.

$$S_\alpha^z(j)S_\alpha^z(j+1) = 1 \qquad \alpha = 2, \ldots, N. \tag{9}$$

With condition (9), equation (8) becomes

$$-\hat{H} = \sum_{j} (\lambda\sigma_1^z(j)\sigma_1^z(j+1) + \sigma_1^x(j)) \tag{10}$$

which shows that there is an Ising-like transition separating the completely ordered (Baxter) phase from that partially ordered one in which just the mean values of the S variables (products of σ) are different from zero. The universal properties of the Hamiltonian (8) are independent of Δ and N, since $\Delta > 1$.

This result can be extended to negative values of K by introducing a variable $\zeta(\boldsymbol{r})$ which is equal to 1 (-1) in any point of the sublattice A (B), as indicated in figure 2. In terms of

$$\eta_\alpha(\boldsymbol{r}) = \zeta(\boldsymbol{r})\sigma_\alpha(\boldsymbol{r}) \tag{11}$$

the Hamiltonian (1) becomes

$$-\frac{\mathcal{H}}{kT} = \sum_{\boldsymbol{r}} \bigg(\sum_{\alpha=1}^{N} [-K^{x}\eta_\alpha(\boldsymbol{r})\eta_\alpha(\boldsymbol{r}+\hat{x}) - K^{\tau}\eta_\alpha(\boldsymbol{r})\eta_\alpha(\boldsymbol{r}+\hat{\tau})] + \tfrac{1}{2}\sum_{\alpha\neq\beta} [K_4^{x}\eta_\alpha(\boldsymbol{r})\eta_\alpha(\boldsymbol{r}+\hat{x})\eta_\beta(\boldsymbol{r})\eta_\beta(\boldsymbol{r}+\hat{x}) + K_4^{\tau}\eta_\alpha(\boldsymbol{r})\eta_\alpha(\boldsymbol{r}+\hat{\tau})\eta_\beta(\boldsymbol{r})\eta_\beta(\boldsymbol{r}+\hat{\tau})]\bigg) \tag{12}$$

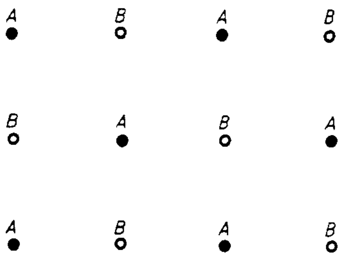

Figure 2. The sublattices A and B where the variable ζ assumes the values 1 and -1, respectively.

which repeats, in the region $K<0$, the critical behaviour already described in the ferromagnetic region. It is worth noticing that, in view of equation (11), the ferromagnetic ordering of the variable η corresponds to an antiferromagnetic array of σ.

Finally, we examine the GW conjecture about the disorder point. We begin by writing the transfer matrix of the N-colour AT model as

$$T = T_2 T_1$$

where

$$T_2 = \exp\left(\sum_j \left[\sum_{\alpha=1}^{N} K^x \sigma_\alpha^z(j)\sigma_\alpha^z(j+1) + \tfrac{1}{2}\sum_{\alpha\neq\beta} K_4^x \sigma_\alpha^z(j)\sigma_\alpha^z(j+1)\sigma_\beta^z(j)\sigma_\beta^z(j+1)\right]\right) \tag{13a}$$

and

$$T_1 = \prod_j \left(1 + \exp\{-2K^\tau[1+(N-1)\Delta]\} \sum_{\alpha=1}^{N} \sigma_\alpha^x(j) + \tfrac{1}{2}\exp\{-4K^\tau[1+(N-2)\Delta]\} \sum_{\alpha\neq\beta} \sigma_\alpha^x(j)\sigma_\beta^x(j) + \ldots + \exp(-2NK^\tau) \prod_{\alpha=1}^{N} \sigma_\alpha^x(j)\right). \tag{13b}$$

Next, we take the continuous time limit, now defined by

$$\exp\{-2K^\tau[1+(N-1)\Delta]\} = \tau \tag{14a}$$

and

$$K^x = \lambda\tau \tag{14b}$$

and find

$$-\hat{H} = \sum_j \left(\sum_{\alpha=1}^{N} [\lambda\sigma_\alpha^z(j)\sigma_\alpha^z(j+1) + \sigma_\alpha^x(j)] + \tfrac{1}{2}\sum_{\alpha\neq\beta} \lambda\Delta\sigma_\alpha^z(j)\sigma_\alpha^z(j+1)\sigma_\beta^z(j)\sigma_\beta^z(j+1)\right) \tag{15}$$

which can be mapped onto a Gross–Neveu model after the fermionisation of each colour (α). However, equation (14b) is consistent with $\tau\to 0$ only if $\Delta > \Delta_c = -1/(N-1)$, where Δ_c is exactly the conjectured location of the disorder point. In fact, at that point other meaningful features occur, the most important being the non-vanishing of the entropy per site at $T=0$. Although we do not know the entropy per site in an exact way we can exhibit a lower bound for it, namely

$$s > k\ln(1+N) \tag{16}$$

which in turn is incompatible with any kind of long-range ordering in the system (Wannier 1950). Indeed, the several ground states of the system, at $T=0$ and $\Delta=\Delta_c$, can be separated by intermediate regions which have the same energy as do the ordered ones. So, long-range ordering does not offer energy advantages to this system.

In summary, we have shown that the N-colour AT model exhibits Ising-like critical behaviour along the line BC, which separates phases III (completely broken symmetry) and II (partially broken symmetry). The result was extended to the antiferromagnetic region and the conjecture of Grest and Widom (about the disorder point) was seen to be in agreement with a Wannier-type argument.

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